Unstable excited and stable oscillating gap 2π pulses

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Dynamics of the gap 2π pulse dynamics in one-dimensional resonantly absorbing Bragg gratings are studied. A new family of stable oscillating and excited unstable gap 2π pulses is analytically and numerically described by a transition from the two-wave Maxwell–Bloch equation to the modified sine-Gordon equation and by direct integration of the two-wave Maxwell–Bloch equation. © 2002 Optical Society of America

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In recent years there has been vast interest in nonlinear pulse propagation in photonic bandgap structures, or photonic crystals.[1] It has been shown[2–4] that, as a result of nonlinear light–matter interaction, an intense laser pulse can propagate at a frequency within the linear forbidden Bragg gap band through a structure with different types of nonlinearity; this is the so called gap soliton. A steady gap soliton moves in a periodic structure as an optical soliton does in a homogeneous medium, keeping its shape and constant velocity. However, the existence of the photonic bandgap gives rise to specific properties of the gap soliton dynamics; for instance, the pulse can stand with zero velocity[2,5,6] or oscillate periodically, changing its amplitude and the sign of its velocity.[7,8] These oscillations have been investigated in the framework of the generalized massive Thirring model for gap solitons in periodic cubic materials.[8] However, oscillations of the gap 2π pulse of self-induced transparency in a resonant periodic structure were demonstrated only numerically in the case of a complicated set of equations in complex functions,[7] and the physical nature of the oscillations has not been described. Physically, it is clear that the oscillations are caused by the photonic bandgap. If an optical soliton is formed by an arbitrary pulse in a homogeneous medium, part of the energy, which was not trapped by the soliton, quickly leaves the region of the slow soliton in a form of free linear radiation. In the gap soliton this untrapped energy is fixed in weakly excited atoms and in a small field that cannot propagate through the structure because of the existence of a linear photonic bandgap. As a result, if the initial soliton velocity is slow enough, the gap soliton that interacts with the perturbation cannot leave the region of the interaction because its kinetic energy is less than the potential energy of the interaction. This blockage gives rise to gap soliton oscillations. In the present paper we study the instability of the gap 2π pulse of self-induced transparency in a resonantly absorbing Bragg grating. It is shown that the initial problem for the simple two-wave Maxwell–Bloch equations in real functions is reduced to a modified sine-Gordon equation. Therefore one can obtain an equation of motion to describe the evolution of the stable oscillating gap 2π pulse and the unstable excited gap 2π pulse, which decays to a steady soliton and a perturbation. The oscillating pulse is physically stable because it does not decay, and it is unsteady because this solution is within a region of oscillatory instability on the phase plane of the equation of motion. Solving a boundary problem, we explain the physical nature of the delayed reflection and the delayed transmission of the gap 2π pulse when the incident pulse forms a gap 2π pulse at low velocity near the boundary.

Let us consider the coherent interaction of light with a one-dimensional resonantly absorbing Bragg grating consisting of periodically distributed thin layers of two-level oscillators. This model closely corresponds to a real structure of periodically arranged quantum wells with resonance excitons in a semiconductor.[9,10] In the exact Bragg condition, the problem of light–matter interaction in a semiclassical approximation is described by the coupled-mode two-wave Maxwell–Bloch (TWMB) equations[2] for the slowly varying envelope of the electric field amplitudes of forward and backward waves $E^{\pm}$, dimensionless polarization P, and the population difference density of two-level oscillators n:

$$\Omega_t{}^{+} + \Omega_x{}^{+} = P,$$

$$\Omega_t{}^{-} - \Omega_x{}^{-} = P,$$

$$P_t = n(\Omega^{+} + \Omega^{-}),$$

$$n_t = -P(\Omega^{+} + \Omega^{-}), \tag{1}$$

where $\Omega^{\pm} = 2\tau_c(\mu/\hbar)E^{\pm}$, $\tau_c = (8\pi\epsilon T_1/3c\rho\lambda^2)^{1/2}$ is the cooperative time that characterizes the mean photon lifetime in the medium preceding resonant absorption,[2] T_1 is the excited level lifetime, ϵ is the dielectric constant of the medium, ρ is the density of resonant oscillators, λ is the wavelength, c is the velocity of light, μ is the matrix element of the transition dipole moment, and $x = x'/c\tau_c$ and $t = t'/\tau_c$ are dimensionless space and time coordinates, respectively.

Using the solution of the Bloch equation $P = -\sin\theta$, where θ is the Bloch angle, we reduce Eqs. (1) to the form

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$$\tilde{\Omega}_x + \Omega_t = -2\sin\theta,$$

$$\Omega_x + \tilde{\Omega}_t = 0,$$

$$\theta_t = \Omega, \tag{2}$$

where $\Omega \equiv \Omega^+ + \Omega^-$ and $\tilde{\Omega} \equiv \Omega^+ - \Omega^-$. The second of Eqs. (2) yields

$$\tilde{\Omega}(x, t) = -\theta_x(x, t) + f(x). \tag{3}$$

Then Eqs. (2) give the following equation for the Bloch angle:

$$\theta_{xx} - \theta_{tt} = 2\sin\theta + f_x(x). \tag{4}$$

This is the modified sine-Gordon equation, for which the function $f(x)$ is determined by the initial condition in Eq. (3):

$$f(x) = \tilde{\Omega}(x, 0) + \theta_x(x, 0). \tag{5}$$

Thus, if the fields and the population inversion are absent in a medium at $t = 0$, i.e., $\tilde{\Omega}(x, 0) = 0$ and $\theta(x, 0) = 0$, or if the steady 2π pulse propagates through the structure and $\tilde{\Omega}(x, t) = -\theta_x(x, t)$,[2] then $f(x) = 0$, and Eq. (4) is reduced to the exact sine-Gordon equation that describes the steady gap 2π pulse. In the general case $f(x) \neq 0$, i.e., if there is a deviation from the exact gap 2π pulse solution in initial condition (5), the gap soliton dynamics will become more complicated. The second term on the right-hand side of Eq. (4) describes the interaction of the soliton of the exact sine-Gordon equation with a localized perturbation and determines the dynamics of gap 2π pulse oscillations and instability.

To solve Eq. (4) we use a simple energetic method,[11] which allows one to obtain an equation of motion for the soliton of the modified sine-Gordon equation when the shape of the modified equation solution is close to the shape of the exact sine-Gordon equation solution; i.e., function $f(x)$ is assumed to be small. Rewriting Eq. (4) in variables $\eta = \sqrt{2}x$, $\tau = \sqrt{2}t$, and $f' = f/\sqrt{2}$, we get the equation in a traditional form:

$$\theta_{\eta\eta} - \theta_{\tau\tau} = \sin\theta + f_\eta'(\eta). \tag{6}$$

The Lagrangian density function for Eq. (6) is

$$L = \tfrac{1}{2}\theta_\tau^{\,2} - \tfrac{1}{2}(\theta_\eta - f')^2 - (1 - \cos\theta),$$

and the corresponding Hamiltonian density is

$$H = \tfrac{1}{2}\theta_\tau^{\,2} + \tfrac{1}{2}\theta_\eta^{\,2} - f'\theta_\eta + \tfrac{1}{2}f'^2 + (1 - \cos\theta). \tag{7}$$

The first four terms on the right-hand side of Eq. (7) fix the energy density $[(\Omega^+)^2 + (\Omega^-)^2]/2$ of the forward and backward waves in the structure.

Because the system is conservative, the total energy of the localized solutions is the integral of motion, $(\mathrm{d}/\mathrm{d}\tau)\int_{-\infty}^{\infty} H\mathrm{d}\eta = 0$; then Eq. (7) yields

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\int_{-\infty}^{\infty}\mathrm{d}\eta\left[\frac{1}{2}\theta_\tau^{\,2} + \frac{1}{2}\theta_\eta^{\,2} + (1 - \cos\theta)\right] = \frac{\mathrm{d}}{\mathrm{d}\tau}\int_{-\infty}^{\infty}\mathrm{d}\eta f'\theta_\eta. \tag{8}$$

Assuming that the shape of the unsteady solution of Eq. (6) is close to that of the soliton of the exact sine-Gordon equation, we write the desired solution for the 2π pulse as

$$\theta = 4\tan^{-1}\left(\exp\left\{\frac{-\eta + \xi(\tau)}{[1 - u^2(\tau)]^{1/2}}\right\}\right), \tag{9}$$

where $u(\tau)$ is the time-dependent soliton velocity and $\xi(\tau) = \int_0^\tau u(\tau')\mathrm{d}\tau'$ is the coordinate of the soliton center. The overlap integral on the right-hand side of Eq. (8) fixes the potential energy of interaction of the kink [Eq. (9)] with the perturbation. Substituting Eq. (9) into Eq. (8) and assuming that u^2 is small, we find the following Newton equation of motion for the coordinate of the pulse center:

$$\xi_{\tau\tau} = -\frac{1}{4}\frac{\partial}{\partial\xi}\int_{-\infty}^{\infty}\mathrm{sech}(\eta - \xi)f'(\eta)\mathrm{d}\eta. \tag{10}$$

Let us take the perturbation [Eq. (5)] in a simple form, $f'(\eta) = f_0\,\mathrm{sech}(\eta)$, which coincides with the shape of the fields $\Omega^\pm$ in the exact 2π pulse solution.[2] Then Eq. (10) gives the following equation of motion of the pulse:

$$\xi_{\tau\tau} = -U_\xi, \qquad U = \frac{f_0}{2}\frac{\xi}{\sinh\xi}. \tag{11}$$

Equation (11) describes the motion of a unit mass quasi-particle in potential U subject to the action of the potential force, $-U_\xi$. Note that generally the potential energy of interaction U_{in} is

$$U_{\mathrm{in}} = \frac{1}{4}\int_{-\infty}^{\infty}\mathrm{sech}(\eta - \xi)f'(\eta)\mathrm{d}\eta. \tag{12}$$

The total energy of the particle [Eq. (11)] is $u^2/2 + U = $ constant; therefore finite motion is possible if the potential is attractive, $f_0 < 0$, and the pulse velocity is sufficiently small on the potential well bottom $|u(\xi = 0)| < \sqrt{-f_0}$ [Fig. 1(a)]. This gap 2π pulse oscillates but does not decay; it is unsteady but stable. An increase in the initial velocity leads to escape of the pulse from the poten-

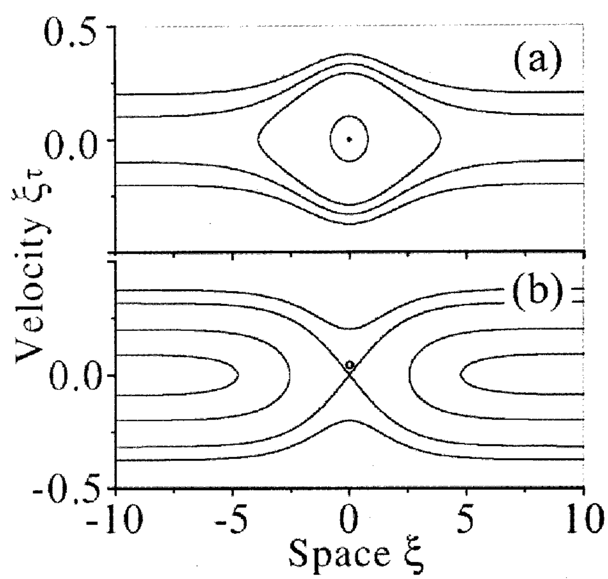

Fig. 1. Phase plane of Eq. (11) for (a) the attractive potential U at $f_0 = -0.1$ and (b) the repulsive potential at $f_0 = 0.1$.

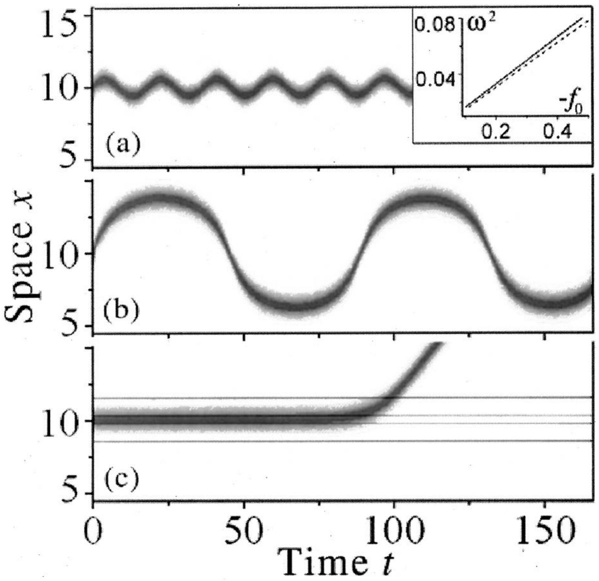

Fig. 2. Evolution of the initial gap 2π pulse (the gray scale is proportional to n). The initial conditions at $t = 0$ are fixed by $n = -\cos\theta$, $P = -\sin\theta$, $\theta = 4\tan^{-1}\exp\lfloor\sqrt{2}(-x + x_0)/\sqrt{1 - {u_0}^2}\rfloor$, and $\Omega^{\pm} = {\Omega_0}^{\pm}\,\mathrm{sech}\lfloor\sqrt{2}(-x + x_0)/\sqrt{1 - {u_0}^2}\rfloor$, where u_0 is the initial pulse velocity. (a) ${\Omega_0}^{+} = 1.45$ and ${\Omega_0}^{-} = -0.87$ correspond to $f_0 = -0.4$ and $u_0 = 0.2$; (b) ${\Omega_0}^{+} = 2.34$, ${\Omega_0}^{-} = -0.48$; $f_0 = -0.4$, and $u_0 = 0.55$; (c) $f_0 = 0.07$ and $u_0 \approx 0$. The contour lines correspond to the perturbation $f(x, t)$ calculated from Eq. (3) for the initial condition $f(x) = \sqrt{2} f_0 \mathrm{sech}\lfloor\sqrt{2}(-x + x_0)/\sqrt{1 - {u_0}^2}\rfloor$. Inset, square of the frequency of pulse harmonic oscillations as a function of perturbation obtained by analytical calculation from Eq. (14) (solid curve) and by numerical integration of Eqs. (1) (dashed curve).

tial well. If initial condition (5) corresponds to positive $f_0 > 0$, the potential of interaction U is positive, and the pulse is repelled from the potential [Fig. 1(b)]. This means that the pulse is unstable and can be described as an excited gap 2π pulse, because it decays to the perturbation and to the steady soliton whose kinetic energy equals the energy of excitation (potential energy of interaction).

Solving Eq. (11), one obtains the law of motion $\xi(\tau)$ in integral form:

$$\int_0^{\xi} \frac{\mathrm{d}\xi'}{(\alpha - f_0\xi'/\mathrm{sh}\,\xi')^{1/2}} = \tau, \tag{13}$$

where $\alpha = u^2(\xi = 0) + f_0$. If displacement of the soliton is small compared with the width of potential $\xi \ll 1$ and $f_0 < 0$, the soliton executes a harmonic motion:

$$\xi = \xi_0 \sin\omega\tau, \qquad \omega^2 = -f_0/6. \tag{14}$$

The value of the frequency of oscillations is in a good agreement with the results of numerical integration of Eq. (1) [see Fig. 2(a), inset]. In a closer approximation, Eq. (10) is reduced to the form

$$\xi_{\tau\tau} - \frac{f_0}{6}\xi + \frac{7f_0}{180}\xi^3 = 0.$$

Oscillatory solutions of this equation are the distinctly anharmonic Jacobian elliptic functions.[12]

As follows from Eqs. (2) and (3), the expressions for the forward and backward wave amplitudes are given by

$$\Omega^{+} = 1/2(\Omega + \tilde{\Omega}) = 1/2[\theta_t - \theta_x + f(x)],$$
$$\Omega^{-} = 1/2(\Omega - \tilde{\Omega}) = 1/2[\theta_t + \theta_x - f(x)], \tag{15}$$

respectively. Substituting solution (9) into Eqs. (15) and into the Bloch equation solutions, and assuming that ξ_t is small, we get the desired approximate solutions for TWMB equations (1):

$$\Omega^{+} = (\xi_t + \sqrt{2})\mathrm{sech}[\sqrt{2}x - \xi(t)] + f(x)/2,$$
$$\Omega^{-} = (\xi_t - \sqrt{2})\mathrm{sech}[\sqrt{2}x - \xi(t)] - f(x)/2,$$
$$n = -\cos\theta = -1 + 2\,\mathrm{sech}^2[\sqrt{2}x - \xi(t)],$$
$$P = -2\,\mathrm{sech}[\sqrt{2}x - \xi(t)]\tanh[\sqrt{2}x - \xi(t)]. \tag{16}$$

In general, the value $\xi(t)$ is found from Eq. (10), and in our special example [Eq. (11)] $\xi(t)$ can be a periodic finite or infinite function (Fig. 1). Note that the amplitudes of fields $\Omega^{\pm}$ [Eqs. (16)] and their sum Ω depend on the velocity $u = \xi_t$, but the difference $\tilde{\Omega}$ does not depend on u.

The energetic method that we used above is convenient but approximate, because one does not take into account the change of the kink's shape. Thus it is important to check our analytical results by direct numerical integration of Eq. (1). Figures 2(a) and 2(b) illustrate the dynamics of the gap 2π pulse in the case of a relatively large amplitude of the attractive potential [Eq. (11)], $f_0 = -0.4$, for several initial pulse velocities. The results were obtained by numerical integration of the initial problem of Eqs. (1) under condition (5): $f(x) = \sqrt{2} f_0\,\mathrm{sech}\lfloor\sqrt{2}(-x + x_0)/\sqrt{1 - {u_0}^2}\rfloor$, where u_0 is the initial pulse velocity. The small velocity u_0 gives rise to harmonic oscillations of the 2π pulse about the attractive potential [Fig. 2(a)]. The shape of the pulse differs only slightly from that of the soliton of the exact sine-Gordon equation. The difference between the analytically [Eq. (14)] and the numerically calculated values of the oscillation frequency is small [Fig. 2(a), inset]. Note that there is good agreement with the analytical results, although the values of f_0 and ξ are not small. By increasing the initial pulse velocity one gets the anharmonic oscillations [Fig. 2(b)] that correspond to the anharmonic branch on the phase plane in Fig. 1(a). In Fig. 2(c) the initial pulse amplitudes ${\Omega_0}^{\pm}$ are chosen such that condition (5) corresponds to the repulsive potential $f(0) = \sqrt{2}f_0 = 0.1$ and the initial velocity is close to zero. Therefore the solution starts its evolution from the point of unstable equilibrium, which is near the center point in Fig. 1(b). This initial gap 2π pulse is excited and unstable because the pulse fields $|{\Omega_0}^{\pm}| = |{\Omega_s}^{\pm} \pm 0.05|$ are larger than the fields in the steady standing gap 2π pulse $|{\Omega_s}^{\pm}| = |\pm\sqrt{2}|$. The sum of the fields, $\Omega = 0$, coincides with the steady solution but the difference of the fields $\tilde{\Omega}_0 = 2\sqrt{2} + 0.1$ is larger. This explains the pulse instability. After some delay, the excited gap 2π pulse decays to a steady moving soliton with velocity $u = 0.26$ ($u^2 = 0.07 = f_0$) and to the standing perturbation [Fig. 2(c)].

Formally, this decay is described as a repulsion of the initial solitonlike solution by the positive potential under the initial condition of unstable equilibrium [Fig. 1(b)].

The analytical analysis of the gap 2π pulse instability in the initial problem described above allows us to explain the dynamics of the gap 2π pulse formation near the structure boundary, which is a realistic physical process. Let us consider the boundary problem of the incident field's interaction with the structure by solving Eqs. (1) numerically with the boundary conditions

$$\Omega^{+}(x = 0, t) = \Omega_0 \operatorname{sech}[(t - t_0)/\tau_0],$$

$$\Omega^{-}(x = l, t) = 0, \qquad \Omega^{\pm}(x, t = 0) = 0,$$

$$n(x, t = 0) = -1, \qquad P(x, t = 0) = 0,$$

where τ_0 is the duration of the incident pulse and l is the structure length. Results of the numerical simulation for various values of incident field amplitude Ω_0 are presented in Fig. 3. If the incident field is large, the steadily moving gap 2π pulse is formed after nonlinear reflection of some part of the incident field [Fig. 3(a)]. Decreasing the value of Ω_0 leads to a more complicated dynamics. Figure 3(b) demonstrates the delayed transmission of the incident pulse. The almost standing excited unstable gap 2π pulse with the very small positive velocity $u \approx 0$ is localized in the structure. The depth of its penetration is sufficient that the boundary influence can be neglected, so it can be treated similarly to the unstable pulse in the initial problem. Sum field Ω is equal to sum Ω_s of the steady gap 2π pulse, but the difference field is larger: $\tilde{\Omega}/\tilde{\Omega}_s = 1.00035$. Therefore there is a small repulsive potential [Eq. (12)]. The corresponding unstable initial state is marked by a circle on the phase plane in Fig. 1(b). After a delay, the excited gap 2π pulse decays to the moving steady gap 2π pulse and to the residual perturbation, i.e., small standing waves and an inversion.

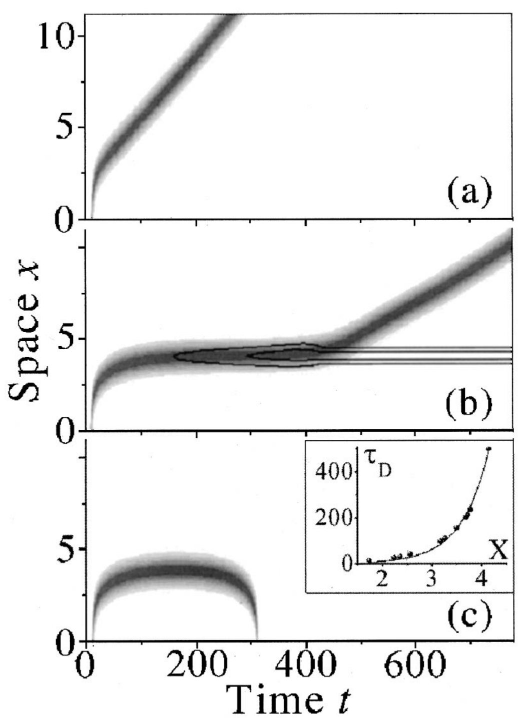

Fig. 3. Evolution of the incident pulse in the structure (the gray scale is proportional to n). Pulse duration, $\tau_0 = 0.84$; amplitudes Ω_0 are (a) 2.701, (b) 2.70063, and (c) 2.70062. The contour lines in (c) show the perturbation $f(x, t)$ calculated from Eq. (3). Inset, dependence of time delay τ_D on the depth of pulse penetration X.

Numerical simulation shows that, if the amplitude of the incident pulse is small, the influence of the boundary on the process of the gap 2π pulse formation is significant. During the process of nonlinear reflection, the small negative attractive potential [Eq. (12)] that corresponds to negative function $f(x)$ is formed in the area between the pulse center and the boundary. Because of this attraction, the pulse cannot propagate into the structure and is reflected [Fig. 3(c)]. The value of time delay τ_D of the incident pulse, in this delayed reflection process, depends exponentially on the depth of the pulse penetration and may be 2 orders of magnitude greater than the duration of the incident pulse. This fact may be interesting for practical applications.

In summary, it has been shown that besides the traditional moving and standing gap 2π pulses,[2,5,7] which are steady soliton solutions, there is a class of stable oscillating and unstable excited gap 2π pulses. A mathematical reason for the existence of this class of pulses lies in the structure of the TWMB equations, which have an additional degree of freedom because of the backward Bloch field and can be transformed into the exact sine-Gordon equation only when there are special initial conditions. In a general case of arbitrary initial conditions, the TWMB equations demonstrate new coupled and unstable solutions. Physically, the oscillations and instability arise as a result of the interaction of the slow gap 2π pulse with localized small-field and excited two-level oscillators. The structure of the periodic layers of two-level systems discussed here is a good model for the realistic periodic structure of quantum wells $In_{0.04}Ga_{0.96}As/GaAs$.[9,10] If the resonant exciton density is 1.7×10^{12} cm^{-3}, the dipole moment of transition is $\mu = 9 \times 10^{-29}$ Cm, the wave length is 830 nm, and the cooperative time is $\tau_c = 0.3$ ps, then to observe the oscillating gap 2π pulse one needs an energy of the incident laser pulse of $\sim$1.3 μJ/cm^2 for a pulse duration of 0.34 ps.

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